INVESTIGACIÓN REVISTA MEXICANA DE FÍSICA **52** (1) 70–73 FEBRERO 2006

Uniformly accelerated observers in special relativity

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Recibido el 25 de octubre de 2005; aceptado el 9 de enero de 2006

The red shift for an electromagnetic wave measured by two observers in a uniformly accelerated frame, which, according to the equivalence principle, should correspond to a gravitational red shift, is calculated as well as the bending of light rays.

Keywords: Special relativity; red shift; bending of light rays.

Se calcula el corrimiento al rojo para una onda electromagnética medido por dos observadores en un sistema de referencia uniformemente acelerado, el cual, de acuerdo con el principio de equivalencia, debe corresponder a un corrimiento al rojo gravitacional, así como la desviación de rayos de luz.

Descriptores: Relatividad especial; corrimiento al rojo; desviación de rayos de luz.

PACS: 03.30.+p; 04.20.-q

1. Introduction

Just like in Newtonian mechanics, in the special theory of relativity the inertial frames of reference play an essential role (see, for example, Refs. 1,2) and the Lorentz transformations give the relationship between the space-time coordinates of events measured in two different inertial frames. However, in Newtonian mechanics, as well as in special relativity, one can make use of non-inertial reference frames. In fact, the equivalence principle states that, in the absence of gravitational fields, a reference frame that is linearly accelerated is locally identical to a reference frame at rest in a gravitational field. In particular, making use of the equivalence principle, it is possible to derive the existence of a gravitational red shift for electromagnetic waves and of a bending of the light rays by considering their propagation viewed from an accelerated frame in the absence of a gravitational field (see, for example, Refs. 3-5). References 3 and 4 contain computations of these effects based on Newtonian relations.

The aim of this paper is to find, in the context of special relativity, the red shift for an electromagnetic wave, measured by two observers in a uniformly accelerated frame separated by a fixed distance, and the trajectory of a light ray. In Sec. 2 we derive the relationship between the space-time coordinates of events measured by an inertial frame and a reference frame whose origin has a constant acceleration with respect to an instantaneously co-moving inertial frame (see also Refs. 6-8). This coordinate transformation is employed in Sec. 3 to find the exact red shift formula and the bending of light rays.

2. Observers with constant acceleration

From the elementary Lorentz transformation formulae for two inertial frames in the standard configuration,

$$t' = \gamma(t - vx/c^2), \quad x' = \gamma(x - vt),$$

where v is the velocity of the inertial frame S$'$ with respect to the inertial frame S and $\gamma = (1 - v^2/c^2)^{-1/2}$, one obtains the transformation formula for the acceleration of a particle

$$a'_x = \frac{(1 - v^2/c^2)^{3/2}}{(1 - vu_x/c^2)^3}\, a_x,$$

where $u_x = \mathrm{d}x/\mathrm{d}t$ is the x-component of the velocity of the particle and $a_x = \mathrm{d}u_x/\mathrm{d}t$. Therefore, for a particle that has a constant acceleration $a'_x = g$ with respect to an inertial frame S$'$ which instantaneously accompanies the particle (*i.e.* $u_x = v$),

$$a_x = (1 - u_x^2/c^2)^{3/2} g. \tag{1}$$

Thus, assuming that $u_x = 0$ for $t = 0$,

$$u_x = \frac{gt}{\sqrt{1 + (gt/c)^2}}. \tag{2}$$

The proper time, τ_0, of the accelerated particle is

$$\tau_0 = \int \sqrt{1 - u_x^2/c^2}\, \mathrm{d}t = \int \frac{\mathrm{d}t}{\sqrt{1 + (gt/c)^2}} = \frac{c}{g} \operatorname{arcsinh}(gt/c) \tag{3}$$

and the position

$$x = \frac{c^2}{g}\sqrt{1 + (gt/c)^2}, \tag{4}$$

if the integration constant is conveniently chosen (see Eq. (6) below). According to Eqs. (3) and (4), the world-line of a particle with constant acceleration, parameterized by its proper time, is given by

$$ct = \frac{c^2}{g}\sinh(g\tau_0/c), \quad x = \frac{c^2}{g}\cosh(g\tau_0/c). \tag{5}$$

Equations (4) or (5) yield

$$x^2 - (ct)^2 = (c^2/g)^2, \tag{6}$$

hence the name of hyperbolic motion for the motion of a particle with constant acceleration (see, for example, Ref. 2, Sec. 3.8).

The tangent vector of the world-line (5)

$$(U^0, U^1) \equiv \left(\frac{\mathrm{d}(ct)}{\mathrm{d}\tau_0}, \frac{\mathrm{d}x}{\mathrm{d}\tau_0}\right) = c\Big(\cosh(g\tau_0/c), \sinh(g\tau_0/c)\Big), \tag{7}$$

points along the time axis for an observer moving along this world-line, and

$$(B^0, B^1) \equiv \Big(\sinh(g\tau_0/c), \cosh(g\tau_0/c)\Big), \tag{8}$$

is a unit space-like vector orthogonal to (U^0, U^1) (in the sense that $(B^0)^2 - (B^1)^2 = -1$, and $B^0U^0 - B^1U^1 = 0$) that defines the spatial direction for this observer (see, for example, Ref. 9). Hence, the parametric equations

$$ct = \frac{c^2}{g}\sinh(g\tau_0/c) + h\sinh(g\tau_0/c),$$
$$x = \frac{c^2}{g}\cosh(g\tau_0/c) + h\cosh(g\tau_0/c), \tag{9}$$

obtained by adding the vector $h(B^0, B^1)$ to the space-time coordinates (5), correspond to the world-line of a second observer ahead of the first by a distance h, measured by the first observer. (Note, however, that the parameter τ_0 appearing in Eqs. (9) is not the proper time of the second observer if $h \neq 0$. See the discussion in the following section.)

Equations (9) also correspond to a hyperbolic motion, in fact

$$x^2 - (ct)^2 = \left(\frac{c^2}{g} + h\right)^2, \tag{10}$$

but the second observer has the constant acceleration $g/(1 + gh/c^2)$ with respect to an inertial frame that instantaneously accompanies the second observer [*cf.* Eq. (6)].

Thus, two uniformly accelerated observers (or particles) must have *different* accelerations in order to remain separated by a proper constant distance. The increasing velocities of the two particles produce an increasing Lorentz contraction in such a way that the two particles seem to approach each other viewed from the inertial frame S, which corresponds to a difference between the accelerations of the particles.

By construction, the events with space-time coordinates

$$\frac{c^2}{g}\Big(\sinh(g\tau_0/c), \cosh(g\tau_0/c)\Big)$$

and

$$\left(\frac{c^2}{g} + h\right)\Big(\sinh(g\tau_0/c), \cosh(g\tau_0/c)\Big)$$

with respect to the inertial frame S [see Eqs. (5) and (9)] are *simultaneous* for an observer moving along the world-line (5) or (9). Now we construct a reference frame with coordinates ct', x', whose origin coincides with the first accelerated observer O [which follows the world-line (5)]; the space-time coordinates ct', x', of any event P are defined in such a way that x' is the spatial distance from the event P to the first observer O, as measured by this observer, and t' is the value that the proper time of the first observer has simultaneously (with respect to O) with the occurrence of P. Thus,

$$ct = \left(\frac{c^2}{g} + x'\right)\sinh(gt'/c),$$
$$x = \left(\frac{c^2}{g} + x'\right)\cosh(gt'/c), \tag{11}$$

which amounts to

$$x \pm ct = \left(\frac{c^2}{g} + x'\right)\exp(\pm gt'/c). \tag{12}$$

Therefore, the Minkowski line element $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2$ takes the form

$$\mathrm{d}s^2 = \left(1 + \frac{gx'}{c^2}\right)^2 c^2\mathrm{d}t'^2 - \mathrm{d}x'^2. \tag{13}$$

(Note that if gx'/c^2 and gt'/c are small, Eqs. (11) reduce to the approximate expressions

$$t \simeq t', \quad x \simeq \frac{c^2}{g} + x' + \frac{1}{2}gt'^2,$$

which agree with the Newtonian formulae.)

Equations (12) imply that, for a light ray propagating along the x-axis, $(c^2/g + x')\exp(\mp gt'/c) = \mathrm{const.}$, where the upper [resp. lower] sign corresponds to rays propagating in the positive [resp. negative] x-direction. Hence, the total time (measured by the clock at O) employed by a light signal to go from O to a point with $x' = h$ and back to O is equal to

$$\frac{2c}{g}\ln(1 + gh/c^2),$$

which amounts to a mean velocity

$$\frac{gh}{c\ln(1 + gh/c^2)} \geq c.$$

The existence of this effect was also considered by Einstein in Ref. 3 and employed in finding the bending of light rays in a gravitational field.

Making use of Eqs. (12), one can readily obtain the inverse relations to Eqs. (11)

$$ct' = \frac{c^2}{2g}\ln\left(\frac{x + ct}{x - ct}\right), \quad x' = \sqrt{x^2 - (ct)^2} - \frac{c^2}{g}. \tag{14}$$

3. Applications

3.1. Red shift

At the origin of the reference frame S′ ($x' = 0$), Eq. (13) reduces to $\mathrm{d}s^2 = c^2\mathrm{d}t'^2$, thus showing that t' is indeed the proper time, τ_0, measured by a clock placed at that point. Similarly, Eq. (13) implies that the time τ_1 measured by a clock fixed with respect to the accelerated frame S′ at $x' = h$, is related to t' by $\mathrm{d}\tau_1 = (1 + gh/c^2)\mathrm{d}t'$ and therefore,

$$\mathrm{d}\tau_1 = \left(1 + \frac{gh}{c^2}\right)\mathrm{d}\tau_0. \tag{15}$$

Thus, by contrast with the case of an inertial frame, the clocks fixed with respect to S′ cannot be synchronized.

The *exact* relation (15) is equivalent to the approximate one derived by Einstein [3] (see also Ref. 4), though, in the present case, g is only the acceleration of the clock at the origin. The difference of accelerations of the two clocks is necessary in order for Eq. (15) to be symmetric; that is, from Eq. (15) we obtain

$$\begin{aligned}\mathrm{d}\tau_0 &= \left(1 + \frac{gh}{c^2}\right)^{-1}\mathrm{d}\tau_1 = \frac{c^2}{c^2 + gh}\mathrm{d}\tau_1\\ &= \left(1 + \frac{g}{1 + gh/c^2}\frac{(-h)}{c^2}\right)\mathrm{d}\tau_1,\end{aligned}$$

which is of the form (15) with the roles of the clocks interchanged, and $g/(1 + gh/c^2)$ being the acceleration of the clock at $x' = h$ [see Eq. (10)].

Equation (15) implies that the frequencies of a light signal, for example, ν_0 and ν_1, respectively, measured by observers at $x' = 0$ and $x' = h$, are related by

$$\nu_0 = \left(1 + \frac{gh}{c^2}\right)\nu_1$$

(*cf.* Refs. 3,4).

3.2. Bending of the light rays

Now we want to find the trajectory followed by a ray of light with respect to the accelerated frame S′. To this end, we note that the Cartesian coordinates of an event in the directions perpendicular to that of the relative motion of S and S′ are related by

$$y = y', \quad z = z', \tag{16}$$

if the Cartesian axes of S and S′ are parallel. With respect to the inertial frame S, a light ray perpendicular to the acceleration of S′ can be represented by the parametric equations

$$x = x_0, \quad y = y_0 + ct, \quad z = 0,$$

where x_0 and y_0 are constants. Then, according to Eqs. (11) and (16),

$$\begin{aligned}x_0 &= \left(\frac{c^2}{g} + x'\right)\cosh(gt'/c),\\ y' &= y_0 + \left(\frac{c^2}{g} + x'\right)\sinh(gt'/c),\\ z' &= 0.\end{aligned}$$

Hence, $x_0^2 - (y' - y_0)^2 = (c^2/g + x')^2$, that is

$$\left(x' + \frac{c^2}{g}\right)^2 + (y' - y_0)^2 = x_0^2,$$

which means that the trajectory of the light ray with respect to the accelerated frame is an arc of the circle centered at $(-c^2/g, y_0, 0)$ with radius $|x_0|$. If the light ray passes through the origin of S′, we must to choose $x_0 = c^2/g$. For $g = 9.8$ m/s^2, the radius c^2/g is approximately equal to 1 light-year. (The light ray traverses only half of the circle with $x' > -c^2/g$ if $x_0 > 0$.)

In a similar way one finds that the trajectory of any light ray with respect to S′ is a straight line parallel to the x'-axis or a circle centered at some point with $x' = -c^2/g$; in the latter case, if the ray passes through the origin of S′, the radius of the circle must be greater than or equal to c^2/g.

This result also follows from the fact that the light rays are null geodesics of the space-time metric, which, in terms of the coordinates of S′, has the form

$$\mathrm{d}s^2 = \left(1 + \frac{gx'}{c^2}\right)^2 c^2\mathrm{d}t'^2 - \mathrm{d}x'^2 - \mathrm{d}y'^2 - \mathrm{d}z'^2 \tag{17}$$

[*cf.* Eq. (13)]. Hence, the time taken by a light ray to go from a point P_1 to another point P_2 is given by

$$\frac{1}{c}\int_{\mathrm{P}_1}^{\mathrm{P}_2} \frac{\sqrt{\mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2}}{1 + gx'/c^2}$$

and, by virtue of Fermat's principle of least time (see, for example, Ref. 10), the light ray follows a geodesic of the three-dimensional metric

$$\frac{\mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2}{(1 + gx'/c^2)^2} \tag{18}$$

(see also Ref. 11), which is the metric of the three-dimensional hyperbolic space (see, for example, Ref. 12). The geodesics of metric (18) are known to be circles whose planes are perpendicular to $1 + gx'/c^2 = 0$ centered at points with $1 + gx'/c^2 = 0$ or straight lines parallel to the x'-axis.

In order to compare the bending of the light rays found in this subsection with the well-known result obtained in the framework of general relativity by means of the Schwarzschild metric [1,2,9], we can find (*approximately*) the radius of curvature of the trajectory of a light ray at the point of closest approach to the mass producing the gravitational field. The curvature radius, R, of a plane curve defined by the equation $r = r(\theta)$, at a point where $\mathrm{d}r/\mathrm{d}\theta$ vanishes, is given by

$$\frac{1}{R} = \frac{1}{r} - \frac{1}{r^2}\frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} = \frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u, \tag{19}$$

with $u \equiv 1/r$. On the other hand, making use of the usual expression for the Schwarzschild metric, the trajectory of a light ray in the gravitational field produced by a mass M is given by the differential equation [2,9]

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{3GM}{c^2}u^2,$$

where G is Newton's constant of gravitation and $u = 1/r$. Thus, owing to Eq. (19), taking into account that r is approximately the radial distance, the radius of curvature of the trajectory of a light ray at the periastron is $R = c^2 r^2/(3GM) \simeq c^2/(3g)$, where g is the acceleration of gravity at a distance r of the mass M, according to Newton's law of gravitation, which *differs* by a factor of 1/3 from the radius obtained above.

Thus, by contrast with the good agreement between the red shift found in the preceding subsection and that given by the general theory of relativity, there is a notorious difference in the case of the bending of a light ray, in spite of the relativistic character of both approaches and especially of the origin of Einstein's theory of gravitation.

Acknowledgments

The authors wish to thank Dr. G. Silva Ortigoza and the referee for valuable suggestions. One of the authors (C.I.P.S.) thanks the Vicerrectoría de Investigación y Estudios de Posgrado of the Universidad Autónoma de Puebla for financial support through the programme "La ciencia en tus manos."

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