Some comments on unitarity gauge

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A pedagogical discussion on the unitarity gauge within the context of Hamiltonian path integral formalism is presented. A model based on the group $O(N)$, spontaneously broken down to the subgroup $O(N-1)$, is used to illustrate the main aspects of this gauge–fixing procedure. Among the issues, discussed with some extent, are: (1) the structure of model's constraints following the Dirac's method, (2) the gauge–fixing procedure, using the unitarity gauge for the massive gauge fields and the Coulomb one for the massless gauge fields, (3) the absence of BRST symmetry in this gauge–fixing procedure and its implications on the renormalizability of the theory, and (4) the static role of the ghost and anti–ghost fields associated with the massive gauge fields and how their contributions can be eliminated by using the dimensional regularization scheme.

Keywords: Unitarity gauge; Hamiltonian path integra.

Se presenta una discusión pedagógica de la norma unitaria en el contexto de integral de trayectoria hamiltoniana. Un modelo basado en el grupo $O(N)$, roto espontáneamente al subgrupo $O(N-1)$, es usado para ilustrar los aspectos principales de este procedimiento de fijación de la norma. Entre los temas, discutidos con cierta extensión, están: (1) la estructura de las constricciones del modelo siguiendo el método de Dirac; (2) el procedimiento de fijación de la norma, usando la norma unitaria para los campos de norma masivos y la norma de Coulomb para los campos de norma sin masa, (3) la ausencia de la simetría BRST en este procedimiento de fijación de la norma y sus implicaciones sobre la renormalizabilidad de la teorí; a y (4) el papel estático de los campos fantasma asociados con los campos de norma masivos y cómo sus contribuciones pueden ser canceladas usando el esquema de regularización dimensional

Descriptores: Norma unitaria; integral de trayectoria hamiltoniana.

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1. Introduction

Gauge theories play a central role in the formulation of modern physics theories. The known fundamental interactions of the nature are all governed by this class of theories. The main feature of a gauge system is that it is formulated using more degrees of freedom than those indeed necessary to describes it. The gauge degrees of freedom, the redundant ones, arise as a consequence of local transformations, called gauge transformations. In the Lagrangian framework, this redundancy leads to the well–known Noether identities [1], while in the Hamiltonian framework, it appears as constraints on the phase space [2, 3]. As a consequence of this gauge freedom, there are many solutions of the equations of motion consistent with the initial data, the system is degenerate in this sense. Though it is possible in principle to eliminate the gauge degrees of freedom, it is not convenient, mainly to preserve manifest covariance and also by calculational convenience. Therefore, it is convenient to keep the gauge degrees of freedom as true dynamical variables and introduce anticommuting fields to cancel their effects in physical observables.

The quantization of gauge systems is not always straightforward, since it is necessary to lift the degeneration through some gauge–fixing procedure. It results that in the Hamiltonian framework it is not possible to define a covariant gauge–fixing procedure, so one ends up with a noncovariant generating functional. In order to recover manifest covariance, one needs to recur to the Faddeev–Popov method [4]. Though this method works well in Yang–Mills theories for a wide class of gauge–fixing procedures, it fails in more general gauge systems. For example, reducible gauge systems, in which the gauge generators are not all independent [5]. There are also gauge systems in which the structure constants depend on the fields [6] or open systems, in which the commutator of two gauge transformations give rise to a trivial gauge transformation which is proportional to the equations of motion [5]. In the least years, a powerful tool based in the antifields Batalin–Fradkin–Vilkovisky formalism [7] has been developed to quantize in a covariant way this class of systems. In this formalism, the generalized BRST transformations play a central role. This formalism has been developed in both the Lagrangian [5] and Hamiltonian [8] framework and their equivalence was proved perturbatively [9]. Yang–Mills theories can be quantized using this general scheme, but since they are of the irreducible type, closed and their structure constants do not depend on fields, *i.e*, their gauge algebra is a Lie algebra, the Faddeev–Popov method is valid. In order to maintain our discussion as simple as possible, we will use this method in studying the unitarity gauge.

Gauge systems can possesses a finite number of degrees of freedom, but those with infinitely many ones (a field theory) are the most interesting from the physical point of view. Doubtless, the simplest and best known field theory, which represents a gauge system, is quantum electrodynamics (QED). This theory is described by the Abelian gauge group $U_e(1)$. At energies higher than the Fermi scale ($v = 246$ GeV), the weak and electromagnetic interactions are unified by means of the Yang–Mills group $SU_L(2) \times U_Y(1)$, known as the electroweak group, where $U_Y(1)$ is the so–called hypercharge group. The standard model (SM) of the strong and electroweak interactions is based in the group $SU_C(3) \times SU_L(2) \times U_Y(1)$, being $SU_C(3)$ the group of the strong interactions.

One peculiarity of the weak interaction is to be mediated by massive gauge bosons. As it is known, a mass term for a gauge boson can not be introduced explicitly in the theory otherwise gauge symmetry is lost. In order to generate masses for these fields, it is necessary to break this symmetry, not explicitly, but spontaneously. This means that the action remains invariant under the gauge group but not the minimal energy state. To do this, it is necessary to introduce scalar fields, in some appropriate representation of the gauge group, that leads to an infinitely degenerate vacuum (the minimal energy state). In this situation, when one choose one specific vacuum, the phenomenon known as spontaneously symmetry breaking (SSB) arises, which means that the vacuum is not invariant under the group. In most physical interest cases, the vacuum is invariant only under a subgroup of the original group, *i.e.* only certain generators of the group do not leave invariant the vacuum, they are broken generators in this sense. When a global invariant theory is considered, there arise massless scalar fields, one for each broken symmetry, known as Goldstone bosons [10]. Though interesting, it is unlikely that massless scalar particles exist in the nature. However, when SSB is combined with local gauge invariance, a new phenomenon arises: the gauge boson fields associated with the broken generators acquire masses. This phenomenon is known as the Higgs mechanism [11]. In the local gauge invariant scheme, the massless scalars do not represent physical degrees of freedom, but they can be removed of the theory in a specific gauge, called the unitarity gauge. The main goal of this work is to study this gauge–fixing procedure both at the classical and quantum levels.

In this paper we present a pedagogical study of the main properties of the unitarity gauge. For this purpose we will use a toy model defined by the orthogonal group $O(N)$, spontaneously broken down to the subgroup $O(N-1)$. Though we first will present a brief study on the Goldstone theorem [10] and the Higgs mechanism [11] in the context of this model, our main purpose is to discuss some peculiarities that arise when one quantize the theory using a gauge–fixing procedure based in the unitarity gauge. In contrast with renormalizable gauge–fixing procedures (R_ξ–gauges) [12], defined by using gauge–fixing functions that depend on gauge and scalars fields, the unitarity gauge is defined using supplementary conditions depending only on the pseudo-Goldstone bosons (PGB). The unitarity gauge is widely used to evaluate tree–level S matrix elements, though it is not necessarily the most appropriate for practical loop calculations. Perhaps, its most important property is that, due to the absence of PGB in the theory, it provides an appropriate scheme to probe unitarity of the S matrix.

Our presentation is organized as follows. In Sec. 2. we discuss the Goldstone theorem and the Higgs mechanism in the context of the $O(N)$ group spontaneously broken down to the $O(N-1)$ subgroup. We will take advantage of this discussion to present the notation and conventions that will be used through the paper. Section 3. is dedicated to study with some detail the structure of the constraints of the model, including the definition of a gauge–fixing procedure. We will lift the degeneration in the massive gauge sector by using the unitarity gauge, while in the massless gauge sector we will introduce the Coulomb gauge. Our main contribution is given in Sec. 4., where the quantization of the theory is presented starting out from the fundamental Hamiltonian path integral. The process of integrating out the generalized momenta as well as the implementation of the gauge–fixing procedure are discussed with some extent. Several aspects arising from the unitarity gauge are discussed. Among other, we show how the unitarity gauge leads to a generating functional which is not invariant under the BRST symmetry. This fact has non-trivial implications on the renormalizability of the theory. We discuss also the role played by the ghost fields associated with the massive gauge fields. In particular, it is show that their contribution can be eliminated if it is used the dimensional regularization scheme [13]. Finally, in Sec. 5. we present our conclusions.

2. The model

The model that will be used to discuss the unitarity gauge is based in the orthogonal group in N dimensions, $O(N)$. Firstly, we discuss the SSB of this group. Let ϕ be an N–component real field, with Lagrangian given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - V(\phi^a), \tag{1}$$

where

$$V(\phi^a) = \frac{1}{2}m^2\phi^a\phi^a + \frac{1}{4}\lambda(\phi^a\phi^a)^2. \tag{2}$$

$\mathcal{L}$ is invariant under global transformations of this group. In this representation, the elements of $O(N)$ are $N \times N$ orthogonal matrices given by $O = \exp\{i\theta^\alpha T^\alpha\} = 1 + i\theta^\alpha T^\alpha + O(\theta^2)$. So, the infinitesimal transformations are given by $\delta\phi^a = i\theta^\alpha (T^\alpha)_{ab}\phi^b$, where T^α are the matrices representing to the $\frac{1}{2}N(N-1)$ generators of the group. Since the representation is real, iT^α must be a real matrix, so T^α is imaginary, and because it is Hermitean, it is antisymmetric. Occasionally, we will need to make explicit use of these matrices. To this end, it is convenient to label these matrices with two indices, as follows $(T^{rs})_{ab}$, with $r \neq s$ denoting the rotation plane and a, b the element of the corresponding matrix. Then, in general, we can write

$$(T^{rs})_{ab} = -i(\delta^r_a\delta^s_b - \delta^s_a\delta^r_b). \tag{3}$$

We now turn to analyze the classical configuration of minimal energy of the system. This configuration corresponds to a field $\phi \equiv \phi_0$, which minimize the potential $V(\phi^a)$, *i.e.* ϕ_0 is solution of the equation

$$\frac{\partial V}{\partial \phi^a} = [m^2 + \lambda(\phi^b\phi^b)]\phi^a = 0. \tag{4}$$

There are two cases. (i) If $m^2 > 0$, the potential has a minimum at $\phi_0 = 0$. In this case, the parameter m can be interpreted (in the context of the quantum theory) as the masses of the fields ϕ^a. (ii) If $m^2 < 0$, the potential has a local maximum at $\phi_0 = 0$ and a minimum at $\phi_0^a\phi_0^a = -\frac{m^2}{\lambda} \equiv v^2$. The most interesting situation is the last one. In this case, the minimal energy state (the vacuum state) is infinitely degenerate, all configuration of fields lying on the hypersphere with radius v satisfy Eq. (4). But these configurations are not arbitrary at all, since all points on this surface are related one to the other through an orthogonal transformation of $O(N)$. SSB occurs when one choose only one of this vacuums. Let us choose the Nth component of ϕ to be the one which develops a vacuum expectation value. We denote the vacuum by the following vector:

$$\phi_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v \end{pmatrix}, \tag{5}$$

(in quantum theory, where the fields ϕ^a are operators, it is said that the field ϕ^N develops the vacuum expectation value $\langle 0|\phi^N|0\rangle = v$). It is clear that this vacuum is not invariant under the complete group $O(N)$, since there exist matrices $O \in O(N)$ which do not leave invariant ϕ_0, *i.e.* $\phi_0' = O\phi_0 \neq \phi_0$. However, there exist a subgroup of $O(N)$ which leaves invariant ϕ_0, namely the subset of rotations about the Nth axes, which form the subgroup $O(N-1)$. Let O' be an element of $O(N-1)$, then $\phi_0' = O'\phi_0 = \phi_0$, with $O' = \exp\{i\theta^{\alpha'}T^{\alpha'}\}$, being the $T^{\alpha'}$ the $\frac{1}{2}(N-1)(N-2)$ generators of $O(N-1)$. Before showing the invariance of ϕ_0 under this subgroup, let us to establish our notation. Through the paper we will use the following conventions:

$$\begin{aligned} a, b, c, \cdots &= 1, \cdots, N \\ \alpha, \beta, \gamma, \cdots &= 1, \cdots, \frac{1}{2}N(N-1) \\ a', b', c', \cdots &= 1, \cdots, (N-1) \\ \alpha', \beta', \gamma', \cdots &= N, \cdots, \frac{1}{2}(N-1)(N-2). \end{aligned} \tag{6}$$

In addition, we will reserve the middle letters of the Greek alphabet, $\mu, \nu, \cdots$ to denote Lorentz indices. As usual, spatial indices will be denoted by the letters $i, j, \cdots$. With this notation, we can conveniently separate the generators of $O(N-1)$ from the remaining ones of $O(N)$ in the way $T^\alpha = (T^{a'}, T^{\alpha'})$. We now show that the vacuum is invariant under the subgroup $O(N-1)$. What $O(N-1)$ leaves invariant ϕ_0 means that it is annihilated by the generators $T^{\alpha'}$. In fact, using Eqs.(3, 5), we obtain

$$(T^{\alpha'\beta'}\phi_0)_a = v(T^{\alpha'\beta'})_{aN} = -iv(\delta_a^{\alpha'}\delta_N^{\beta'} - \delta_N^{\alpha'}\delta_a^{\beta'}) = 0,$$

as it is evident from the fact that $\alpha' \neq \beta'$ and from Eq.(6).

Making the shift $\phi = \varphi + \phi_0$, the Lagrangian (1) takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi^a)(\partial^\mu\varphi^a) - \lambda(v\varphi^N + \frac{1}{2}\varphi^a\varphi^a)^2 - \frac{1}{4}\lambda v^4. \tag{7}$$

Evidently, φ^N is the field of a particle with mass $(2\lambda v^2)^{1/2}$ while the $N-1$ fields $\varphi^{a'}$ are massless. Notice that the number of massless bosons is the same as the number of broken generators $T^{a'}$. This is a special case of a general result known as Goldstone theorem [10], which establish that for each broken generator there is a massless scalar field, called Goldstone boson. In our example, the group $O(N)$ has $\frac{1}{2}N(N-1)$ symmetries, while the subgroup $O(N-1)$, which leaves invariant the vacuum, has $\frac{1}{2}(N-1)(N-2)$ symmetries. The number of broken symmetries is then $\frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1$, which coincides with the number of Goldstone bosons. To conclude this part, let us to emphasize that the Lagrangian (7) is still invariant under the $O(N)$ group. It must be so since nothing violating this symmetry has been introduced. However, it is important to stress that the vacuum ϕ_0 is not invariant under the complete group $O(N)$, but only under the subgroup $O(N-1)$. The group $O(N)$ is spontaneously broken to the $O(N-1)$ subgroup in this sense.

We now proceed to discus the same theory analyzed above but when it is invariant under local gauge transformations of the $O(N)$ group. It is unlikely that Goldstone bosons exist in the nature. However, when SSB and gauge invariance are combined, there is an exception to Goldstone theorem. This combination is known as the Higgs mechanism [11], which play a fundamental role in the description of the SM and its extensions.

The gauge invariant version of the Lagrangian (1) is given by

$$\mathcal{L} = \frac{1}{2}(D_\mu^{ab}\phi^b)(D_{ac}^\mu\phi^c) - V(\phi^a) - \frac{1}{4}F_{\mu\nu}^\alpha F_\alpha^{\mu\nu}, \tag{8}$$

where $D_\mu^{ab} = \delta^{ab}\partial_\mu - igT_{ab}^\alpha A_\mu^\alpha$ is the covariant derivative in the N–dimensional representation of $O(N)$, A_μ^α are the gauge fields, and g is the coupling constant. The Yang–Mills tensor $F_{\mu\nu}^\alpha$ is given by

$$F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + gf^{\alpha\beta\gamma}A_\mu^\beta A_\nu^\gamma, \tag{9}$$

where $f^{\alpha\beta\gamma}$ are the structure constants of the group. Under infinitesimal local transformations, the matter and gauge fields transform respectively as

$$\delta\phi^a(x) = i\theta^\alpha(x)(T^\alpha)_{ab}\phi^b(x), \tag{10}$$

$$\delta A_\mu^\alpha(x) = \frac{1}{g}\mathcal{D}_\mu^{\alpha\beta}(x)\theta^\beta(x), \tag{11}$$

where $\mathcal{D}^{\alpha\beta}_\mu = \delta^{\alpha\beta}\partial_\mu - gf^{\alpha\beta\gamma}A^\gamma_\mu$ is the covariant derivative in the adjoint representation of $O(N)$.

Since the potential $V(\phi^a)$ in this Lagrangian is the same that the one appearing in the globally invariant Lagrangian (1), it is clear that the gauge invariant theory presents SSB when $m^2 < 0$. We now investigate the consequences of considering SSB and gauge invariance together. After SSB, *i.e.* after making the shift $\phi = \varphi + \phi_0$, the Lagrangian (8) takes the form

$$\mathcal{L} = \frac{1}{2}(D_\mu\varphi)_a(D^\mu\varphi)_a + (D_\mu\phi_0)_a(D^\mu\varphi)_a + \frac{1}{2}(D_\mu\phi_0)_a(D^\mu\phi_0)_a - \frac{1}{4}\lambda(2v\varphi^N - \varphi^a\varphi^a)^2 - \frac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha, \quad (12)$$

where an irrelevant constant term has been ignored. As in the globally invariant case, φ^N is the field of a particle of mass $(2\lambda v^2)^{1/2}$ while the remaining $N-1$ components of ϕ are massless. There is, however, a new ingredient, the third term of this Lagrangian. Taking into account that $T^{\alpha'}\phi_0 = 0$, this term can be written as

$$\frac{1}{2}(D_\mu\phi_0)_a(D^\mu\phi_0)_a = -\frac{1}{2}g^2A^{a'}_\mu A^{b'\mu}(T^{a'}\phi_0)_a(T^{b'}\phi_0)_a = -\frac{1}{2}g^2v^2A^{a'}_\mu A^{a'\mu}, \quad (13)$$

where use of Eq.(3) was made. This shows that $N-1$ of the gauge fields are massive, while the remaining $\frac{1}{2}(N-1)(N-2)$ ones are massless. Notice that the number of massive gauge fields are the same as the number of broken symmetries of $O(N)$ and the number of massless gauge fields coincides with the number of symmetries of the subgroup $O(N-1)$. Notice also that the vector formed by the PGB, $\phi' = (\phi^1, \phi^2, \cdots, \phi^{N-1})$, transform as the $N-1$–dimensional representation of the group $O(N-1)$.

It is a well known fact that a massive gauge field possesses three polarization states, instead of the two ones that characterize the massless gauge fields. Seemingly it seems that we have end up with more degrees of freedom that those of the original theory. This is not the case because of the $N-1$ Goldstone bosons represent spurious degrees of freedom, they can be removed of the theory in a specific gauge. To do this, one writes the ϕ fields in a nonlinear way as follows:

$$\phi = \exp\left\{i\frac{T^{a'}\eta^{a'}}{v}\right\}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ v+h \end{pmatrix}, \quad (14)$$

where the fields $\eta^{a'}$ are the pseudo–Goldstone bosons (PGB) and we have renamed the massive scalar field as $h = \varphi^N$, which is known as the Higgs field. Expanding the above expression at first order in the fields and using Eq.(3) for the elements of the matrices $T^{a'N}$, we recover the linear representation $\phi = (\eta^1, \eta^2, \cdots, \eta^{N-1}, v+h)$. Since the Lagrangian (12) is invariant under $O(N)$, we can make the following special gauge transformation:

$$\phi' = O\phi, \quad (15)$$

$$T^{a'}A'^{a'}_\mu = OT^{a'}A^{a'}_\mu O^\dagger - \frac{1}{g}(\partial_\mu O)O^\dagger, \quad (16)$$

where

$$O = \exp\left\{-i\frac{T^{a'}\eta^{a'}}{v}\right\}. \quad (17)$$

Since $\phi' = (0, 0, \cdots, v+h)$, the PGB disappear when the Lagrangian is written in the new gauge. This is known as the unitarity gauge. In that gauge, the new Lagrangian (12) is obtained by putting all PGB equal to zero: $\mathcal{L}_{UG} = \mathcal{L}|_{\phi^{a'}=0}$. It was shown by Weinberg that unitarity gauge always exists [14]. This gauge–fixing procedure will be discussed in the next section within the context of the Hamiltonian framework by using the Dirac's method for constrained systems.

3. The Hamiltonian of the theory

According to Dirac, to quantize a given system it is necessary to put the theory in the Hamiltonian form. However, the standard method used for regular (not constrained) systems does not works for the case of singular (constrained) systems due to the presence of constraints.

The Hamiltonian formalism for constrained systems was developed by Dirac [2]. The method allows us determine all system's constraints, which define a surface in the complete phase space. The state of the system evolves on this surface, but its evolution may be not unique due to the absence of a well–defined Hamiltonian. When an unique Hamiltonian exists, it is said that the system is subject to second–class constraints. If it is not the case, the system possesses first–class constraints. Gauge systems are subject to first–class constraints, which are intimately related with the gauge symmetry of the theory. In this case, the Dirac's algorithm does not leads to an unique Hamiltonian, it depends on arbitrary Lagrange multipliers. This Hamiltonian describes a degenerate system in the sense that given a state at an initial time, it evolves following many histories on the constraint surface. These histories must be recognized as physically equivalent because they are consequence of arbitrary Lagrange multipliers in the Hamiltonian. At a later time, the corresponding physically equivalent states on the histories form an orbit (see Fig.1). The states on the orbit are related one to another through a gauge transformation, the generators being the first–class constraints. It is clear that only one set of coordinates, corresponding to a representative point of the orbit, is necessary to specify the state of the system at a given time. In order to specify a representative set of variables it is necessary to introduce supplementary conditions, known in the literature as gauge fixing–conditions, which lift the degener-

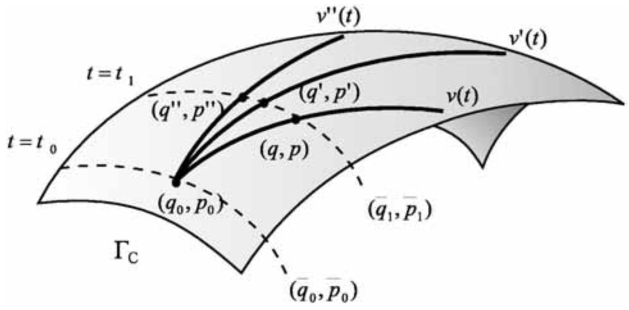

FIGURE 1. Time evolution of a gauge system. Γ_c is the constraint surface defined by all first–class constraints. Continue and dashed lines represent histories and orbits, respectively.

ation of the system. For this aim, the number of gauge–fixing functions must be equal to the number of the first–class constraints. These functions can not be arbitrary at all, they must have nonvanishing Poisson brackets with the first–class constraints, which implies that first–class constraints and gauge–fixing functions together form a set of second–class constraints. Since we can lift the degeneration in many different ways, it is clear that there exist many physically equivalent classical theories, each determined once a specific gauge–fixing procedure has been chosen [3].

We now proceed to illustrate the main aspects of a gauge system by studying the $O(N)$ gauge invariant theory discussed previously. In particular, we have interested in studying those aspects that result of using the unitary gauge to lift the degeneration in the massive gauge sector. Our goal consist in deriving the Hamiltonian for the theory characterized by the Lagrangian given by Eq.(8). The generalized momenta associated with the matter and gauge fields are defined by

$$\pi_a \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}^a} = D_0^{ab}\phi^b, \tag{18}$$

$$\pi_\alpha^0 \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_0^\alpha} = 0, \tag{19}$$

$$\pi_\alpha^i \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_i^\alpha} = F_{i0}^\alpha, \tag{20}$$

(the shift introduced after SSB is not relevant for the present discussion since the definition of the generalized momenta are not affected by it). From the first and third equations we can express the velocities $\dot{\phi}^a$ and $\dot{A}_i^\alpha$ in terms of fields and momenta as follows:

$$\dot{\phi}^a = \pi_a + ig(T^\alpha)_{ab}A_0^\alpha\phi^b, \tag{21}$$

$$\dot{A}_i^\alpha = \pi_i^\alpha + \mathcal{D}_i^{\alpha\beta}A_0^\beta. \tag{22}$$

On the other hand, from the second equation we can see that the $\dot{A}_0^\alpha$ velocities can not be expressed in terms of coordinates and momenta. Instead of this, we have the primary constraints

$$\Phi_\alpha^{(1)} \equiv \pi_\alpha^0 \approx 0, \tag{23}$$

(Through the paper, we will write weak equations using the symbol $\approx$).

The time evolution of the system is governed by the primary Hamiltonian, which is given by

$$\begin{aligned} H^{(1)} &= \int d^3x\mathcal{H}^{(1)} \\ &= \int d^3x(\mathcal{H}_c + \lambda^\alpha\Phi_\alpha^{(1)}), \end{aligned} \tag{24}$$

where λ^α are Lagrange multipliers and $\mathcal{H}_c$ is the canonical Hamiltonian. Using the expressible velocities given by Eqs.(21), we obtain

$$\begin{aligned} \mathcal{H}_c = \frac{1}{2}\pi_a\pi_a + \frac{1}{2}\pi_\alpha^i\pi_\alpha^i + A_0^\alpha[ig(T^\alpha)_{ab}\pi^a\phi^b - \mathcal{D}_i^{\alpha\beta}\pi_\beta^i] \\ -\frac{1}{2}(D_i^{ab}\phi^b)(D_{ac}^i\phi^c) + V(\phi^a) + \frac{1}{4}F_{ij}^\alpha F_\alpha^{ij}. \end{aligned} \tag{25}$$

The primary Hamiltonian depends on the Lagrange multipliers λ^α, which in principle can be determined by demanding consistency conditions on the primary constraints. A basic consistency requirement is the preservation of the constraints in time. So, we demand that

$$\dot{\Phi}_\alpha^{(1)} = \{\Phi_\alpha^{(1)}, H^{(1)}\} \approx 0, \tag{26}$$

where the symbol $\{,\}$ stands for Poisson brackets (PB). In this stage, three different situations can arise. In one them, one can simply obtain the identity $0 = 0$. In this case, the process terminates and the Lagrange multipliers remain undetermined. Another situation arises when one end up with functions of the coordinates and momenta which are independent of both the primary constraints and the Lagrange multipliers. Such functions must be recognized as new constraints and are called secondary constraints. A third situation can occurs when one obtains relations involving the Lagrangian multipliers. In this case, some or all Lagrange multipliers are determined. In our case, we have

$$\begin{aligned} \dot{\Phi}_\alpha^{(1)}(x) = \int d^3y[\{\Phi_\alpha^{(1)}(x), \mathcal{H}_c(y)\} \\ + \lambda^\beta(y)\{\Phi_\alpha^{(1)}(x), \Phi_\beta^{(1)}(y)\}] \approx 0 \\ = \int d^3y\{\Phi_\alpha^{(1)}(x), \mathcal{H}_c(y)\}, \end{aligned} \tag{27}$$

where it is understood that the PB are calculated at equal times. The last expression arises from the fact that the primary constraints only depend on the momenta π_α^0. Then, the consistency condition does not determines the Lagrange multipliers but leads to secondary contraints given by

$$\Phi_\alpha^{(2)} = \mathcal{D}_i^{\alpha\beta}\pi_\beta^i - ig(T^\alpha)_{ab}\pi^a\phi^b \approx 0. \tag{28}$$

Following with the Dirac's algorithm, the secondary constraints must also satisfy consistency conditions similar to the primary ones:

$$\dot{\Phi}^{(2)}_\alpha(x) = \int d^3y[\{\Phi^{(2)}_\alpha(x), \mathcal{H}_c(y)\} + \lambda^\beta(y)\{\Phi^{(2)}_\alpha(x), \Phi^{(1)}_\beta(y)\}] \approx 0$$
$$= \int d^3y\{\Phi^{(2)}_\alpha(x), \mathcal{H}_c(y)\}, \quad (29)$$

where the last expression arises as a consequence from the fact that the PB among the primary and secondary constrainst vanish trivially. The calculation of the last PB is not trivial and some work has to be done. After using the basic properties of the PB together with the equations of motion, we obtain

$$\dot{\Phi}^{(2)}_\alpha = gA^\gamma_0 f^{\alpha\beta\gamma}\Phi^{(2)}_\beta \approx 0. \quad (30)$$

We have the situation $0 = 0$ on the constraint surface. There are no new constraints and the Lagrangian multipliers remain undetermined.

According to Dirac, a function F on the phase space is a first–class quantity if it has, at least weakly, vanishing PB with all the constraints. On the contrary, a function is called a second–class quantity if it has nonvanishing PB with, at least, one of the constraints. This definition allows us to classify the constraints in first– and second–class ones. In our case, all are first–class constraints because

$$\{\Phi^{(1)}_\alpha(x), \Phi^{(1)}_\beta(y)\} = 0, \quad (31)$$

$$\{\Phi^{(2)}_\alpha(x), \Phi^{(1)}_\beta(y)\} = 0, \quad (32)$$

$$\{\Phi^{(2)}_\alpha(x), \Phi^{(2)}_\beta(y)\} = gf^{\alpha\beta\gamma}\Phi^{(2)}_\gamma(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}) \approx 0. \quad (33)$$

In order to define an unique Hamiltonian, it is necessary to introduce supplementary conditions. Before doing this, let us present some brief comments about the role played by the first– class constraints in the dynamics of the theory. As it was mentioned previously, the primary Hamiltonian is the generator of time evolution. Then, consider the time evolution of a function F:

$$\dot{F}(\mathbf{x},t) = \int d^3y[\{F(\mathbf{x},t), \mathcal{H}_c(\mathbf{y},t)\} + \lambda^\alpha(\mathbf{y},t)\{F(\mathbf{x},t), \Phi^{(1)}(\mathbf{y},t)\}]. \quad (34)$$

Let F_0 be the initial value of F at $t = t_0$. At a later time $t > t_0$, we end up with different values for F, depending of the values chooses for the Lagrange multipliers λ^α. In other words, F takes different values on the different histories on which evolves the state of the system. The transformation connecting the values of the function on two different histories, F_{λ^α} and $F_{\lambda'^\alpha}$, is given by a gauge transformation . The infinitesimal variation of F between two near points on the same orbit is given by

$$\delta F = \int d^3y\epsilon^\alpha(\mathbf{y},t)\{F(\mathbf{x},t), \Phi^{(1)}_\alpha(\mathbf{y},t)\}, \quad (35)$$

where $\epsilon^\alpha = \delta t(\lambda^\alpha - {\lambda'}^\alpha)$. This means that the primary first–class constraints are the generators of the gauge transformations. According to Dirac's conjecture, also the secondary first–class constraints might be considered as gauge generators. Though it has been found counterexamples where this conjecture fails [3], it is indeed not the case for Yang–Mills theories, in which is necessary to consider the secondary first–class constraints in order to make the connection with the local gauge transformations introduced at the level of the Lagrangian. To take into account these constraints as generators of gauge transformations, it is necessary to include them in the theory by defining an extended Hamiltonian as follows:

$$H_E = H^{(1)} + \int d^3x\lambda^l_s(x)\Phi^{(s)}_l, \quad (36)$$

where $s = 2, 3\cdots$ stands for secondary, thirdary, $\cdots$ first–class constraints. Usually, secondary, thirdary, $\cdots$ constraints are all called simply as secondary, since there is no physical reasons to distinguish among them. In our case, the extended Hamiltonian can be written as

$$\mathcal{H}_E = \mathcal{H}^{(1)} + u^\alpha\Phi^{(2)}_\alpha$$
$$= \mathcal{H} + \lambda^\alpha_2\Phi^{(2)}_\alpha + \lambda^\alpha\Phi^{(1)}_\alpha, \quad (37)$$

where $\mathcal{H} = \mathcal{H}_c - A^\alpha_0\Phi^{(2)}_\alpha$, since the secondary constraints are already present in the canonical Hamiltonian. Besides, $\lambda^\alpha_2 = A^\alpha_0 + u^\alpha$ and $\lambda^\alpha = \dot{A}^\alpha_0$, which are completely unknown. Taking H_E as the generator of time evolution, we have for the variation of the F function

$$\delta F = \int d^3y\epsilon^\alpha_r(\mathbf{y},t)\{F(\mathbf{x},t), \Phi^{(r)}_\alpha(\mathbf{y},t)\}, \quad (38)$$

where $r = 1, 2$. From this expression, we can see that if F is a first–class quantity, then $\delta F = 0$, *i.e.* F is constant on the orbit. It is a gauge invariant quantity in this sense. This allows us to define a classical observable as those quantity which is a first–class quantity. In particular, $H^{(1)}$ and H_E are first–class quantities. We now apply this equation to the ϕ^a and A^α_μ fields. The corresponding infinitesimal changes are given by

$$\delta\phi^a = -ig\epsilon^\alpha_2(T^\alpha)_{ab}\phi^b, \quad (39)$$

$$\delta A^\alpha_\mu = \epsilon^\alpha_1 - \mathcal{D}^{\alpha\beta}_i\epsilon^\beta_2. \quad (40)$$

If we compare these expressions with the local gauge transformations given in Eqs.(10, 11), we see that they can be reproduced if

$$\epsilon^\alpha_1 = \frac{1}{g}\mathcal{D}^{\alpha\beta}_0\theta^\beta, \quad (41)$$

$$\epsilon^\alpha_2 = -\frac{1}{g}\theta^\alpha. \quad (42)$$

It is clear now that both primary and secondary constraints are needed as gauge generators in order to reproduce the local gauge transformations.

3.1. The gauge–fixing procedure: the unitarity gauge

As it was seen, the Hamiltonian defined above for the $O(N)$ model represents a degenerate theory since it depends on arbitrary Lagrange multipliers. This means that one has indeed many equivalent classical theories, each defined once a gauge–fixing procedure has been chosen to determine the Lagrange multipliers. According to canonical quantization, there is a quantum theory corresponding to each classical Hamiltonian and since all of them are physically equivalent, one may believe that the quantum versions of these classical theories must also be physically equivalent. It is, therefore, reasonable to quantize only one of the physically equivalent classical theories. The gauge–fixing procedure that we will choose is the following. We will lift the degeneration in the massive gauge sector by using the unitarity gauge, which is defined by means of the PGB. In the massless gauge sector, we will use the Coulomb gauge. Then, our gauge–fixing procedure is defined by

$$\chi^{(1)}_{a'} = gv\phi^{a'} \approx 0, \tag{43}$$

$$\chi^{(1)}_{\alpha'} = \partial_i A^i_{\alpha'} \approx 0, \tag{44}$$

where we have introduced the group constant g by convenience and the vacuum expectation value v by dimensional considerations. The first equation define the unitarity gauge, while the second one is called the Coulomb gauge. It is important to notice that it is not possible to introduce at this stage the covariant (Lorenz) gauge $\partial^\mu A^{\alpha'}_\mu$ because it involves the arbitrary velocities $\dot{A}^{\alpha'}_0$. Notice also that, unlike the Coulomb gauge, the unitarity gauge is manifestly covariant.

As in the case of genuine constraints, we demand that the supplementary conditions satisfy consistency conditions

$$\dot{\chi}^{(1)}_{a'} = \{\chi^{(1)}_{a'}, H_E\} \approx 0, \tag{45}$$

$$\dot{\chi}^{(1)}_{\alpha'} = \{\chi^{(1)}_{\alpha'}, H_E\} \approx 0. \tag{46}$$

These conditions lead not to the determination of the Lagrange multipliers, but to new constraints given by

$$\chi^{(2)}_{a'} = gv\pi_{a'} + ig^2 v A^\alpha_0 (T^\alpha)_{a'N}\phi^N \approx 0, \tag{47}$$

$$\chi^{(2)}_{\alpha'} = -\partial_i \pi^i_{\alpha'} - \partial_i \mathcal{D}^{\alpha'\alpha}_i A^\alpha_0 \approx 0. \tag{48}$$

Again, we demand that

$$\dot{\chi}^{(2)}_{a'} = \{\chi^{(2)}_{a'}, H_E\} \approx 0, \tag{49}$$

$$\dot{\chi}^{(2)}_{\alpha'} = \{\chi^{(2)}_{\alpha'}, H_E\} \approx 0. \tag{50}$$

It can be shown that these conditions do not lead to new constraints, but to the determination of the Lagrange multipliers. Instead of presenting this explicitly, we will show that the primary and secondary constraints together with the supplementary conditions given by Eqs.(43, 44, 47, 48) form a set of second–class constraints. This guarantees the determination of the Lagrange multipliers. We need to probe that the matrix formed with all the PB among the constraints is nonsingular. After some algebra, we obtain

$$\begin{pmatrix}
0 & 0 & 0 & 0 & \{\Phi^{(1)}_\alpha, \chi^{(2)}_{a'}\} & \{\Phi^{(1)}_\alpha, \chi^{(2)}_{\alpha'}\} \\
0 & 0 & \{\Phi^{(2)}_\alpha, \chi^{(1)}_{a'}\} & \{\Phi^{(2)}_\alpha, \chi^{(1)}_{\alpha'}\} & \{\Phi^{(2)}_\alpha, \chi^{(2)}_{a'}\} & \{\Phi^{(2)}_\alpha, \chi^{(2)}_{\alpha'}\} \\
0 & \{\chi^{(1)}_{a'}, \Phi^{(2)}_\alpha\} & 0 & 0 & \{\chi^{(1)}_{a'}, \chi^{(2)}_{b'}\} & 0 \\
0 & \{\chi^{(1)}_{\alpha'}, \Phi^{(2)}_\alpha\} & 0 & 0 & 0 & \{\chi^{(1)}_{\alpha'}, \chi^{(2)}_{\beta'}\} \\
\{\chi^{(2)}_{a'}, \Phi^{(1)}_\alpha\} & \{\chi^{(2)}_{a'}, \Phi^{(2)}_\alpha\} & \{\chi^{(2)}_{a'}, \chi^{(1)}_{b'}\} & 0 & 0 & 0 \\
\{\chi^{(2)}_{\alpha'}, \Phi^{(1)}_\alpha\} & \{\chi^{(2)}_{\alpha'}, \Phi^{(2)}_\alpha\} & 0 & \{\chi^{(1)}_{\alpha'}, \chi^{(1)}_{\beta'}\} & 0 & 0
\end{pmatrix},$$

where the nonvanishing elements are given by

$$\{\chi^{(1)}_{a'}(\mathbf{x}), \Phi^{(2)}_\alpha(\mathbf{y})\} = -ig(T^\alpha)_{a'b}\phi^b(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}), \tag{51}$$

$$\{\chi^{(1)}_{\alpha'}(\mathbf{x}), \Phi^{(2)}_\alpha(\mathbf{y})\} = \partial_i \mathcal{D}^i_{\alpha\alpha'}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}), \tag{52}$$

$$\{\chi^{(2)}_{a'}(\mathbf{x}), \Phi^{(1)}_\alpha(\mathbf{y})\} = ig^2 v(T^\alpha)_{a'N}\phi^N(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}), \tag{53}$$

$$\begin{aligned}\{\chi^{(2)}_{a'}(\mathbf{x}), \Phi^{(2)}_\alpha(\mathbf{y})\} &= ig^2 v[(T^\alpha)_{aa'}\pi_a(\mathbf{y}) \\ &\quad - ig(T^\alpha)_{bN}(T^\beta)_{a'N}A^\beta_0(\mathbf{x})\phi^b(\mathbf{y})]\delta(\mathbf{x}-\mathbf{y}),\end{aligned} \tag{54}$$

$$\{\chi^{(2)}_{a'}(\mathbf{x}), \chi^{(1)}_{b'}(\mathbf{y})\} = -g^2 v^2 \delta_{a'b'}\delta(\mathbf{x}-\mathbf{y}), \tag{55}$$

$$\{\chi^{(2)}_{\alpha'}(\mathbf{x}), \Phi^{(1)}_\alpha(\mathbf{y})\} = \partial_i \mathcal{D}^i_{\alpha'\alpha}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}), \tag{56}$$

$$\begin{aligned}\{\chi^{(2)}_{\alpha'}(\mathbf{x}), \Phi^{(2)}_\alpha(\mathbf{y})\} &= gf^{\alpha'\alpha\beta}\pi^i_\beta(\mathbf{y})\partial_i\delta(\mathbf{x}-\mathbf{y}) \\ &\quad + gf^{\alpha'\beta\gamma}\partial_i[A^\beta_0(\mathbf{x})\mathcal{D}^{\alpha\gamma}_i(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})],\end{aligned} \tag{57}$$

$$\{\chi^{(2)}_{\alpha'}(\mathbf{x}), \chi^{(1)}_{\beta'}\} = \partial_i\partial^i\delta_{\alpha'\beta'}\delta(\mathbf{x}-\mathbf{y}). \tag{58}$$

Since neither the rows nor the columns are null, this matrix is nonsingular, though it is important to mention that this is holds only locally since for large values of the fields the Gribov phenomenon arises and no gauge–fixing is possible [15].

4. Quantization

Having defined a gauge–fixing procedure in the previous section, we now turn to quantize the theory. We will use the path integral method, which is the appropriate for Yang–Mills theories. We will start from the Hamiltonian path integral (HPI), which is the fundamental one. Next, we will integrate out the generalized momenta in order to recover an expression defined in the configuration space [the Lagrangian path integral

(LPI)]. In this intermediate stage, some additional terms will arise. We will see that the Coulomb gauge leads to a noncovariant term, while the unitarity gauge is responsible for the presence of a nonpolinomial divergent term in the Higgs field. Manifest covariance is recovered by recurring to the Faddeev–Popov method [4], whereas the divergent term can be eliminated using the dimensional regularization scheme.

4.1. The Hamiltonian path integral

According to Faddeev's theorem [16], the HPI for a system subject to first–class constraints only is given by

$$Z[J]=\int D\phi^a DA^\alpha_\mu D\pi_a D\pi^\mu_\alpha Det\left\{[\chi_\alpha(x),\Phi_\beta(y)]\,\delta(x_0-y_0)\right\}\delta(\Phi^{(1)}_\alpha)\delta(\Phi^{(2)}_\alpha)\delta(\chi^{(1)}_{a'})\delta(\chi^{(1)}_{\alpha'})\delta(\chi^{(2)}_{a'})\delta(\chi^{(2)}_{\alpha'})$$
$$\times\exp\left[i\int d^4x\left(\pi_a\dot{\phi}^a+\pi^\mu_\alpha\dot{A}^\alpha_\mu-\mathcal{H}_c+J\cdot\varphi\right)\right], \quad (59)$$

where $J\cdot\varphi=J_a\phi^a+J^\mu_\alpha A^\alpha_\mu$ are the source terms. In our case, the determinant appearing in the measure of the generating functional can be written as

$$Det[\{\chi_\alpha(x),\Phi_\beta(y)\}]=Det[\{\chi^{(1)}_\alpha(x),\Phi^{(2)}_\beta(y)\}]Det[\{\chi^{(2)}_\alpha(x),\Phi^{(1)}_\beta(y)\}]=(-1)^{N(1-N)/2}Det^2[\{\chi^{(1)}_\alpha(x),\Phi^{(2)}_\beta(y)\}], \quad (60)$$

where the last expression arises after using the identity

$$\{\chi^{(1)}_\alpha(x),\Phi^{(2)}_\beta(y)\}=(-1)^{N(N-1)/2}\{\chi^{(2)}_\alpha(x),\Phi^{(1)}_\beta(y)\}, \quad (61)$$

as it can be verified from the expressions given in Eqs.(51). This determinant can be decomposed as a product of two determinants as follows [17]:

$$Det[\{\chi^{(1)}_\alpha(x),\Phi^{(2)}_\beta(y)\}]=Det[\{\chi^{(1)}_{a'}(x),\Phi^{(2)}_{b'}(y)\}]Det[\{\chi^{(1)}_{\alpha'}(x),\Phi^{(2)}_{\beta'}(y)\}]$$
$$=Det^2[g^2v\delta_{a'b'}\phi^N\delta(\mathbf{x}-\mathbf{y})]Det^2[\partial_i\mathcal{D}^i_{\alpha'\beta'}\delta(\mathbf{x}-\mathbf{y})], \quad (62)$$

where the last expression was obtained by using the results given by Eqs.(51). According to our notation, the indices $\alpha',\cdots$ and $a',\cdots$ are used to denote unbroken and broken generators of $O(N)$, respectively [see Eqs.(6)]. Then, the generating functional takes the form

$$Z[J]=\int D\phi^a DA^\alpha_\mu D\pi_a D\pi^i_\alpha\delta(\Phi^{(2)}_\alpha)\delta(\chi^{(1)}_{a'})\delta(\chi^{(1)}_{\alpha'})\delta(\chi^{(2)}_{a'})\delta(\chi^{(2)}_{\alpha'})Det^2[g^2v\delta_{a'b'}\phi^N\delta(x-y)]Det^2[\partial_i\mathcal{D}^i_{\alpha'\beta'}\delta(x-y)]$$
$$\times\exp\left[i\int d^4x\left(\pi_a\dot{\phi}^a+\pi^i_\alpha\dot{A}^\alpha_i-\mathcal{H}_c+J\cdot\varphi\right)\right]. \quad (63)$$

In this expression, a trivial integration on the momenta π^α_0 was carried out. Also, a constant factor independent of fields was ignored, since it not contributes to physical amplitudes. The Dirac deltas can be transformed using the identity

$$\delta(F(\varphi^m))=Det^{-1}\left[\frac{\delta F(x)}{\delta\varphi^m(y)}\right]\delta(\varphi^m-\hat{\varphi}^m), \quad (64)$$

where $\hat{\varphi}^m$ is a solution of $F(\varphi^m)=0$. We have found convenient to transform the following Dirac deltas:

$$\delta(\chi^{(2)}_{a'})=Det^{-1}[g^2v\delta_{a'b'}\phi^N\delta(x-y)]\delta(A^{b'}_0-\hat{A}^{b'}_0), \quad (65)$$

$$\delta(\chi^{(2)}_{\alpha'})=Det^{-1}[\partial_i\mathcal{D}^i_{\alpha'\beta'}\delta(x-y)]\delta(A^{\beta'}_0-\hat{A}^{\beta'}_0). \quad (66)$$

Notice that these expressions cancel the squared in the determinants that appear in the generating functional. After integrating on the $A^\alpha_0=(A^{a'}_0,A^{\alpha'}_0)$ fields, we obtain

$$Z[J]=\int D\phi^a DA^\alpha_i D\pi_a D\pi^i_\alpha Det\left[g^2v\delta_{a'b'}\phi^N\delta(x-y)\right]Det\left[\partial_i\mathcal{D}^i_{\alpha'\beta'}\delta(x-y)\right]\delta(\Phi^{(2)}_\alpha)\delta(\chi^{(1)}_{a'})\delta(\chi^{(1)}_{\alpha'})$$
$$\times\exp\left\{i\int d^4x\left[-\frac{1}{2}\pi_a\pi_a-\frac{1}{2}\pi^i_\alpha\pi^i_\alpha+\pi_a\dot{\phi}^a+\pi^i_\alpha\dot{A}^i_\alpha+\hat{A}^\alpha_0\Phi^{(2)}_\alpha+\frac{1}{2}(D^{ab}_i\phi^b)(D^i_{ac}\phi^c)-V(\phi^a)-\frac{1}{4}F^\alpha_{ij}F^{ij}_\alpha+J\cdot\varphi\right]\right\}. \quad (67)$$

The fields A_0^α can be introduced newly in the generating functional by using the following integral representation of $\delta(\Phi_\alpha^{(2)})$:

$$\delta(\Phi_\alpha^{(2)}) = \int DQ^\alpha \exp\left\{ i \int d^4x Q^\beta \Phi_\beta^{(2)} \right\}, \quad (68)$$

where the Q^α are arbitrary scalar fields. After doing this, we make the change of variables $A_0^\alpha{=}Q^\alpha{+}\hat{A}_0^\alpha{\Rightarrow}DQ^\alpha{=}DA_0^\alpha$. The result is

$$Z[J] = \int D\phi^a DA_\mu^\alpha D\pi_a D\pi_\alpha^i Det\left[g^2 v \delta_{a'b'}\phi^N \delta(x-y)\right] Det\left[\partial_i \mathcal{D}^i_{\alpha'\beta'}\delta(x-y)\right] \delta(\chi_{a'}^{(1)})\delta(\chi_{\alpha'}^{(1)})$$
$$\times \exp\left\{ i \int d^4x \left[-\frac{1}{2}\pi_a\pi_a - \frac{1}{2}\pi_\alpha^i\pi_\alpha^i + \pi_a D_0^{ab}\phi^b + \pi_\alpha^i F_{0i}^\alpha + \frac{1}{2}(D_i^{ab}\phi^b)(D_{ac}^i\phi^c) - V(\phi^a) - \frac{1}{4}F_{ij}^\alpha F_\alpha^{ij} + J\cdot\varphi \right] \right\}, \quad (69)$$

where we have used the identity

$$\pi_a\dot{\phi}^a + \pi_\alpha^i \dot{A}_\alpha^i + A_0^\alpha \Phi_\alpha^{(2)} = \pi_a D_0^{ab}\phi^b + \pi_\alpha^i F_{0i}^\alpha. \quad (70)$$

The integrals on the momenta π_a and π_α^i are of gaussian type and can immediately be solved. Let

$$I[\varphi] = \frac{1}{2}\int d^4x d^4y \varphi(x)A(x,y)\varphi(y) + \int d^4x B(x)\varphi(x) + C$$

be a quadratic functional. Then, the solution of the corresponding gaussian functional integral is given by

$$\int D\varphi \exp\{-I[\varphi]\} = \exp\{-I[\bar{\varphi}]\}(Det A)^{-1/2}, \quad (71)$$

where $I[\bar{\varphi}]$ is the stationary value of I, being $\bar{\varphi}$ the stationary point, which is solution of $(\delta I/\delta\varphi)|_{\bar{\varphi}} = 0$. In our case, we have

$$\int D\pi_a \exp\{-I[\pi_a]\} = \exp\left\{ i\int d^4x \frac{1}{2}(D_0^{ab}\phi^b)(D_{ac}^0\phi^c)\right\}, \quad (72)$$

$$\int D\pi_\alpha^i \exp\{-I[\pi_\alpha^i]\} = \exp\left\{ i\int d^4x \frac{1}{2}F_{0i}^\alpha F_{0i}^\alpha\right\}, \quad (73)$$

where we have dropped the determinants since they are independent of the fields. After using these results, we have

$$Z[J] = \int D\phi^a DA_\mu^\alpha Det[g^2 v\delta_{a'b'}\phi^N\delta(x-y)] \times Det[\partial_i\mathcal{D}^i_{\alpha'\beta'}\delta(x-y)]\delta(gv\phi^{a'})\delta(\partial_i A^i_{\alpha'}) \times \exp\left\{ i\int d^4x[\mathcal{L} + J\cdot\varphi]\right\}, \quad (74)$$

where $\mathcal{L}$ is the gauge invariant Lagrangian given by Eq.(8). Notice, however, that the generating functional is not covariant due to the presence in the measure of integration of the term $Det[\partial_i\mathcal{D}^i_{\alpha'\beta'}\delta(x-y)]\delta(\partial_i A^i_{\alpha'})$. However, our result is a special case of a more general one due to Faddeev and Popov [4], which states that the LPI for a Yang–Mills theory can be written as

$$Z[J] = \int DA_\mu^\alpha Det\left(\frac{\delta f^\alpha}{\delta\theta^\beta}\right)\delta(f^\alpha(A_\mu^\beta)) \exp\left[i\int d^4x(\mathcal{L} + J_\alpha^\mu \cdot A_\mu^\alpha)\right], \quad (75)$$

where the f^α are the gauge–fixing functions, which can be covariant. We can see that our result can be reproduced when $f^{\alpha'} = \partial_i A_i^{\alpha'}$. As was mentioned in the introduction, the Faddeev–Popov method is adequate only in Yang–Mills theories, it fails in more general gauge systems. On the basis of this result, we can replace our noncovariant term by its 4–dimensional version $Det[\partial_\mu\mathcal{D}^\mu_{\alpha'\beta'}\delta(x-y)]\delta(\partial_\mu A^\mu_{\alpha'})$, *i.e*, we can rewrite our generating functional in terms of the Lorentz gauge as follows:

$$Z[J] = \int D\phi^a DA_\mu^\alpha Det[g^2 v\delta_{a'b'}\phi^N\delta(x-y)] \times Det[\partial_\mu\mathcal{D}^\mu_{\alpha'\beta'}\delta(x-y)]\delta(gv\phi^{a'})\delta(\partial_\mu A^\mu_{\alpha'}) \times \exp\left\{ i\int d^4x[\mathcal{L} + J\cdot\varphi]\right\}, \quad (76)$$

which is covariant, since the unitarity gauge is Lorentz invariant. The determinant on the gauge fields can be expressed in terms of a gaussian integral on anticommuting real fields as follows:

$$Det[\partial_\mu\mathcal{D}^\mu_{\alpha'\beta'}\delta(x-y)] = \int Dc^{\alpha'}D\bar{c}^{\alpha'} \times \exp\left\{ i\int d^4x c^{\alpha'}\partial_\mu\mathcal{D}^\mu_{\alpha'\beta'}\bar{c}^{\beta'}\right\}. \quad (77)$$

The Lorentz gauge can be generalized by introducing an auxiliary field $R^{\alpha'}$ in the form $\partial_\mu A^\mu_{\alpha'} - R_{\alpha'} = 0$. Then, we average over $R^{\alpha'}$ with a gaussian weight

$$\int DR^{\alpha'} \exp\left\{ -i\int d^4x \frac{1}{2\xi}R^{\beta'}R^{\beta'}\right\}\delta(\partial_\mu A^\mu_{\alpha'} - R_{\alpha'}) = \exp\left\{ -i\int d^4x \frac{1}{2}(\partial_\mu A^{\alpha'\mu})^2\right\}, \quad (78)$$

where ξ is the so–called gauge parameter. After doing this, we obtain

$$Z[J]=\int D\phi^a DA^\alpha_\mu Dc^{\alpha'} D\bar{c}^{\alpha'}\delta(gv\phi^{a'})Det[g^2 v\delta_{a'b'}\phi^N\delta(x-y)] \times \exp\left\{i\int d^4x\left[\mathcal{L}-\frac{1}{2\xi}(\partial_\mu A^\mu_{\alpha'})^2+c^{\alpha'}\partial_\mu\mathcal{D}^\mu_{\alpha'\beta'}\bar{c}^{\beta'}+J\cdot\varphi\right]\right\}, \quad (79)$$

where now J includes sources for the ghost ($c^{\alpha'}$) and anti-ghost ($\bar{c}^{\alpha'}$) fields. We now implement the unitarity gauge. Instead of proceeding as the Coulomb gauge case, we eliminate the PGB by using explicitly the supplementary condition, that is, we integrate out the PGB, which is trivial due to the presence of the Dirac delta. The result is

$$Z[J]=\int DhDA^\alpha_\mu Dc^{\alpha'} D\bar{c}^{\alpha'} Det\left[\mathcal{M}_{a'b'}\left(1+\frac{h}{v}\right)\delta(x-y)\right] \times \exp\left\{i\int d^4x\left[\mathcal{L}|_{\phi^{a'}=0}-\frac{1}{2\xi}(\partial_\mu A^\mu_{\alpha'})^2+c^{\alpha'}\partial_\mu\mathcal{D}^\mu_{\alpha'\beta'}\bar{c}^{\beta'}+J\cdot\varphi\right]\right\}, \quad (80)$$

where $\mathcal{M}_{a'b'}=g^2v^2\delta_{a'b'}$ is the mass matrix of the $N-1$ gauge fields. It is important to mention that, as consequence of the integration of the PGB fields, an important property of gauge theories has been lost, namely the BRST [18] symmetry underling to any gauge system. We will turn on this later on. For the moment, note that one can express the determinant on the Higgs field in terms of a gaussian integral on anticommuting fields as follows:

$$Z[J]=\int DhDA^\alpha_\mu Dc^{a'} D\bar{c}^{a'} Dc^{\alpha'} D\bar{c}^{\alpha'} \times \exp\left\{i\int d^4x\left[\mathcal{L}|_{\phi^{a'}=0}-\frac{1}{2\xi}(\partial_\mu A^\mu_{\alpha'})^2+c^{a'}\mathcal{M}_{a'b'}\left(1+\frac{h}{v}\right)\bar{c}^{b'}+c^{\alpha'}\partial_\mu\mathcal{D}^\mu_{\alpha'\beta'}\bar{c}^{\beta'}+J\cdot\varphi\right]\right\}. \quad (81)$$

Notice that there are no derivative terms for the anticommuting fields $c^{a'}$ and $\bar{c}^{a'}$, which means that they do not propagate. This is a peculiarity of the unitarity gauge, which will be analyzed in connection with the dimensional regularization scheme in the next subsection.

To conclude this section, let us present some comments concerning the BRST symmetry [18] that remain after lifting the degeneration of a gauge system (the action which is invariant under generalized BRST transformations is a functional on fields and antifields and it is degenerate. The nondegenerate gauge–fixed BRST action, which is invariant under the usual BRST [18] transformations, define the quantum theory and it is obtained from the extended classical action after eliminating the antifields by means of some gauge–fixing procedure [5]). Since gauge–fixed BRST transformations coincide essentially with the infinitesimal gauge transformations, it is clear that the action defining the generating functional in Eq.(81) is not invariant under this symmetry because the PGB have explicitly been removed of the theory. However, if in the generating functional given in Eq.(79) we treat the Dirac delta $\delta(gv\phi^{a'})$ in the same way that it was made for the case of the Coulomb gauge, we arrive at the following generating functional

$$Z[J]=\int D\phi^a DA^\alpha_\mu Dc^{a'} D\bar{c}^{a'} Dc^{\alpha'} D\bar{c}^{\alpha'} \times \exp\left\{i\int d^4x\left[\mathcal{L}-\frac{1}{2\xi}M^2\phi^{a'}\phi^{a'}-\frac{1}{2\xi}(\partial_\mu A^\mu_{\alpha'})^2+c^{a'}\mathcal{M}_{a'b'}\left(1+\frac{h}{v}\right)\bar{c}^{b'}+c^{\alpha'}\partial_\mu\mathcal{D}^\mu_{\alpha'\beta'}\bar{c}^{\beta'}\right]\right\}. \quad (82)$$

Using now the identity

$$\exp\left\{-i\int d^4x\frac{1}{2\xi}f^a f^a\right\} = \int DB^a \exp\left\{i\int d^4x\left[\frac{\xi}{2}B^aB^a+f^aB^a\right]\right\}, \quad (83)$$

we can finally write the generating functional as

$$Z[J]=\int D\phi^a DA^\alpha_\mu Dc^{a'} D\bar{c}^{a'} Dc^{\alpha'} D\bar{c}^{\alpha'} DB^{a'} DB^{\alpha'} \times \exp\{iS_{BRST}\}, \quad (84)$$

where S_{BRST} is the gauge-fixed BRST action, which is given by

$$S_{BRST}=\int d^4x\left[\mathcal{L}+(M\phi^{a'})B^{a'}+\frac{\xi}{2}B^{a'}B^{a'} +c^{a'}\mathcal{M}_{a'b'}\left(1+\frac{h}{v}\right)\bar{c}^{b'}+(\partial_\mu A^{\alpha'\mu})B^{\alpha'} +\frac{\xi}{2}B^{\alpha'}B^{\alpha'}+c^{\alpha'}\partial_\mu\mathcal{D}^\mu_{\alpha'\beta'}\bar{c}^{\beta'}\right]. \quad (85)$$

Here $B^{a'}$ and $B^{\alpha'}$ are BRST invariant auxiliary scalar fields of dimension two. Since this theory is invariant under BRST transformations, their Green's functions must satisfy Slanov–Taylor identities. Though this theory would be renormalizable, its behaviour at one–loop would be quite problematic since it contains bilinear terms that mix massive gauge fields with their PGB. On the other hand, since BRST symmetry is essential to probe renormalizability at the level of Green's functions, it would be clear now why off–shell renormalizabilty does not works with the theory given by Eq.(81). However, it should be emphasized that renormalizability exists at the level of S matrix. Unitarity gauge, as defining the generating functional in Eq.(81), is not manifestly renormalizable in the sense that it is not renormalizable off–shell. The Green's functions generated by the functionals (81) and (84) are quite different.

4.2. Static ghosts and dimensional regularization

Previously, we found that the unitarity gauge leads to the presence of an action for the anticommuting fields associated with the massive gauge fields given by

$$S_{sgh} = \int d^4x \Big[c^{a'} \mathcal{M}_{a'b'} \Big(1 + \frac{h}{v}\Big) \bar{c}^{b'}\Big]. \tag{86}$$

By definition, the propagator of a particle is the inverse of the operator defining the quadratic term of the corresponding field. From Eq.(86) we can see that in this case such operator is given by

$$\Delta(x,y) = M^2 \delta(x-y),$$

with $M = gv$. So, the propagator of the anticommuting fields is simply

$$\Delta^{-1}(x,y) \equiv G(x,y) = \frac{1}{M^2}\delta(x-y),$$

which shows that these fields do not propagate. In order to investigate the implications of this term in perturbation theory, we proceed to analyze it starting from the determinant that appears in Eq.(81). This determinant leads to the following effective action

$$\begin{aligned} \exp\Big\{iS^{eff}_{sgh}\Big\} &= \int Dc^{a'} D\bar{c}^{a'} \exp\Big\{i\int d^4x \Big[c^{a'} \mathcal{M}_{a'b'} \Big(1 + \frac{h}{v}\Big) \bar{c}^{b'}\Big]\Big\} \\ &= Det\Big[M^2\Big(1 + \frac{gMh}{M^2}\Big)\Big] = DetO = \exp\left\{i[-iTr\log O]\right\}, \end{aligned} \tag{87}$$

hence

$$\begin{aligned} S^{eff}_{sgh} &= -iTr\log O = -iTr\log[\Delta(1+\Delta^{-1}(gMh))] \\ &= -iTr\log\Delta - iTr\log[1+\Delta^{-1}(gMh)]. \end{aligned} \tag{88}$$

Here Tr means trace over discrete and continuous indices. The first term in the above expression is irrelevant since it does not depend on the fields. On the other hand, the last term can be expanded in powers of the coupling constant g as follows

$$\begin{aligned} S^{eff}_{sgh} &= -iTr\log[1+\Delta^{-1}(gMh)] \\ &= iTr\sum_{k=1}^{\infty} \frac{(-gM)^k}{k}[Gh]. \end{aligned} \tag{89}$$

Taking into account that $Tr \rightarrow \delta_{a'a'} \int d^4x = (N-1)\int d^4x$, we can write

$$\begin{aligned} S^{eff}_{sgh} &= i(N-1)\bigg\{(-gM)\int d^4x G(x,x)h(x) + \frac{(-gM)^2}{2}\int d^4x d^4y G(x,y)h(y)G(y,x)h(x) + \cdots\bigg\} \\ &= i(N-1)\bigg\{\Big(-\frac{g}{M}\Big)\int d^4x \delta(x-x)h(x) + \frac{1}{2}\Big(-\frac{g}{M}\Big)^2 \int d^x d^4y \delta(x-y)h(y)\delta(y-x)h(x) + \cdots\bigg\} \\ &= i(N-1)\delta(0)\int d^4x \bigg\{\Big(-\frac{g}{M}\Big)h(x) + \frac{1}{2}\Big(-\frac{g}{M}\Big)^2 h^2(x) + \cdots\bigg\} \\ &= -i(N-1)\delta(0)\int d^4x \log\Big(1+\frac{h}{v}\Big). \end{aligned} \tag{90}$$

This nonpolinomial quartically divergent term was firstly derived using the operator canonical formalism in Ref. 14. More recently, it was derived in Ref. 19 from the standard Feynman diagramatic framework. Of course, our functional derivation of this term is equivalent to the diagramatic approach, since each term of the series can be identified with a Feynman diagram (see Fig.2). This term is necessary in order to remove divergencies that arise at one–loop in processes involving an arbitrary number of external Higgs bosons. This fact was verified explicitly in Ref. 20. Nevertheless, this term can be eliminated if one regularize the theory using the dimensional scheme [13]. One important result of this scheme is the following:

$$\int \frac{d^Dk}{(2\pi)^D}\frac{(k^2)^n}{(k^2-R^2)^m} = i\frac{(-1)^{n-m}}{(4\pi)^{D/2}} \times\frac{\Gamma(n+D/2)\Gamma(m-n-D/2)(R^2)^{D/2+n-m}}{\Gamma(D/2)\Gamma(m)}. \quad (91)$$

Using this formula we can see that S^{eff}_{sgh} disappears since $\delta(0)=\delta(x-x)=\int d^Dk/(2\pi)^D=0$.

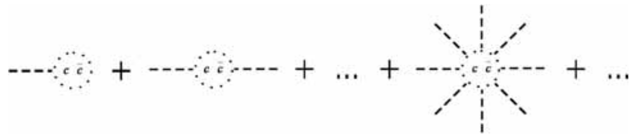

FIGURE 2. Feynman diagrams for the static ghosts contribution to the nonpolinomial term. Dashed and dotted lines represent Higgs bosons and ghosts, respectively.

5. Summary

In this paper we have presented a study of the unitarity gauge within the context of the Hamiltonian path integral formalism. The main features of this gauge–fixing procedure were studied using a toy model based in the group $O(N)$ spontaneously broken to the subgroup $O(N-1)$. At the classical level, the structure of the constraints following the Dirac method, as well as, the definition of a gauge–fixing procedure based in the unitarity and Coulomb gauges were discussed with some detail. At the quantum level, the structure of the generating functional in this gauge–fixing procedure was discussed. It was shown that unitarity gauge leads to a Lagrangian path integral which is not invariant under BRST transformations. It was shown that a BRST invariant generating functional can be constructed, but it involves the PGB in a complicated way, not appropriate to perform practical calculations. Due to the absence of the BRST symmetry, the theory is not renormalizable at the level of Green's functions. It was shown that in this gauge the ghost fields associated with massive gauge bosons do not propagate and that they lead to a quartically divergent nonpolinomial term in the Higgs fields. This term was explicitly calculated using functional techniques and it was shown that it vanishes if the theory is regularized using the dimensional scheme.

We hope this elementary discussion clarifies the study of gauge systems quantization, and the student be motived to attack further problems, real ones, like the standard model or more complicated theories.

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